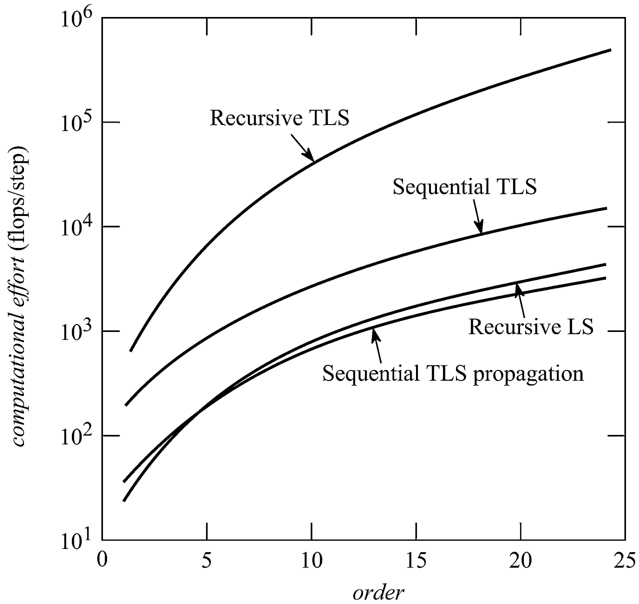

Fig. 1 Averaged computational effort for LS and TLS estimators.

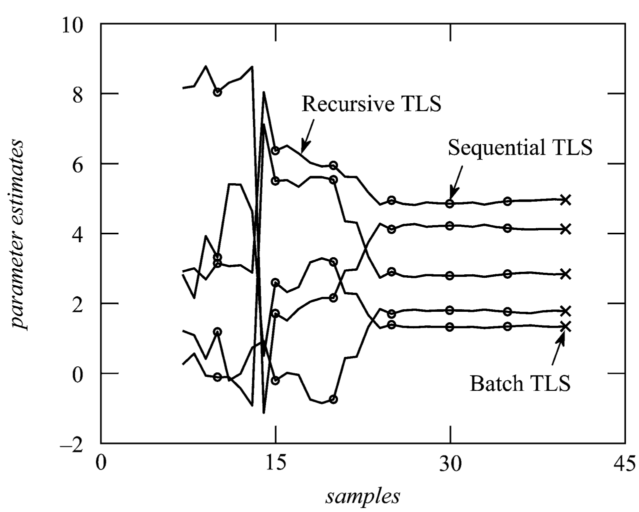

Fig. 2 Resulting estimates for five parameters from 40 samples.

method does not depend on the parameter estimate for its propagation, parameter estimates can be computed at a reduced rate or by another process without loosing numerical accuracy. This is illustrated in Fig. 2, where the estimates for a five-element parameter vector are shown for the various TLS methods. The crosses at the right-hand side mark the results from an SVD-based batch TLS estimation that includes 40 samples; the solid lines indicate the recursive TLS solution that approaches the batch solution with each additional sample that is included. The circles indicate selected estimates with the proposed sequential method, in this case once every five samples. It is clear that sequential TLS provides the same parameter estimates as recursive TLS and that both converge towards the batch solution.

Conclusions

The application of total least squares to typical aerospace parameter estimation problems was briefly discussed. The commonly mentioned threat of information loss by reducing the variables matrix to its inner square was analyzed and found harmless to applications where a series of measurements arrives with time. Together with the notion that instead of singular values, only the smallest eigenvector of the inner square matrix is required to compute TLS estimates this led to the presentation of a computationally superior sequential TLS method.

The suggested method satisfies all of the requirements on an estimator for real-time applications: Its computational demand for each step is independent of the number of preceding measurements and memory requirements are constant. Propagation of the inverted inner square matrix with each arriving measurement does not depend on computation of the actual parameter estimate; without it, the number of operations per step is deterministic and smaller than that for the recursive ordinary least-squares estimator.

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Optimal Interplanetary Trajectories Using Constant Radial Thrust and Gravitational Assists

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Nomenclature

a_r	=	radial thrust acceleration
dr/dt	=	radial rate
$d\theta/dt$	=	angular rate
$E(k, \phi)$	=	elliptic integral of the second kind
$F(k, \phi)$	=	elliptic integral of the first kind
h	=	angular momentum
K_{ACO}	=	total energy after cutoff
K_{BCO}	=	total energy before cutoff
p	=	semilatus rectum
r, θ	=	polar coordinates
r_{CO}, v_{CO}	=	cutoff radius and velocity

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r_f = final orbit apoapse radius
r_0 = radius of the initial assumed circular orbit
T_{BCO} = kinetic energy before cutoff
t_B = burn time
t_T = total time of flight
V_{BCO} = potential energy before cutoff
α = in-plane thrust angle
β = dimensionless radial thrust parameter
Δv_{circ} = change in velocity to circularize trajectory
ε = orbital energy
μ = gravitational constant

Introduction

MINIMAGNETOSPHERIC plasma propulsion (M2P2) is a relatively new form of plasma propulsion. It works by creating a magnetic field around a spacecraft and inflating it with plasma. This "bubble" then interacts with the solar wind creating a constant radial thrust.[1]

Boltz[2] and Prussing and Coverstone-Carroll[3] showed that the maximum radial distance that can be achieved from the central body with continuous constant radial thrust is twice the initial circular radius if the spacecraft must stay on a cyclical orbit that will return it to its initial orbit radius. This result was discouraging to any mission planner wanting to use a radial thrust system for a mission to any planets outside the orbit of Mars.

If, however, the propulsion system is used only for an initial period and then turned off, other cyclical orbits become possible. This Note describes a process to determine thrust duration to achieve a final elliptical orbit around the sun that will have an aphelion as desired and will still pass through the initial circular orbit radius.

Numerical analysis expands the choices of possible missions and optimal trajectories through the addition of a small off-axis thrust angle and gravitational assists. The off-axis thrust angle is made possible by the ability to tilt the magnetic dipole in the M2P2 system, thereby creating lift from the interaction with the solar wind.[1]

Analytic Analysis

For the analytical process, strictly radial thrust is assumed. The first step is to determine the radial distance at which the radial thrust should be terminated. The kinetic energy before cutoff of the vehicle can be expressed as

$$T_{BCO} = \frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] \tag{1}$$

As shown in Eq. (2), the potential energy can be written as the combination of the standard orbital potential along with an added term to account for the radial thrust acceleration,

$$V_{BCO} = -(\mu/r) - a_r r \tag{2}$$

The radial and angular rates can be expressed as

$$\frac{dr}{dt} = \sqrt{(r - r_0)\left[2a_r - \frac{\mu(r - r_0)}{r^2 r_0}\right]} \tag{3}$$

$$\frac{d\theta}{dt} = \frac{h}{r^2} \tag{4}$$

where h is the angular momentum of the orbit. The angular momentum of the system stays constant because the applied thrust is completely radial and, therefore, does not produce a torque that would affect the overall angular momentum. The angular momentum can be written in terms of the semilatus rectum as

$$h = \sqrt{\mu p} \tag{5}$$

The total energy before cutoff of the radial thrust propulsion system is the sum of the two energy terms expressed in Eqs. (1) and (2) and substituting the relations in Eqs. (3) and (4) for the radial and angular velocities,

$$K_{BCO} = -a_r r - \mu/r + \tfrac{1}{2}\left\{p\mu/r^2 + (r - r_0)\left[2a_r - \mu(r - r_0)\big/r^2 r_0\right]\right\} \tag{6}$$

Similarly, the total energy of the spacecraft after the termination of the propulsion system is constant and can be written in terms of the cutoff radius,

$$K_{ACO} = T_{BCO} - \frac{\mu}{r} = -\frac{\mu}{r_{CO}} + \frac{1}{2}\left\{\frac{p\mu}{r_{CO}^2} + (r_{CO} - r_0)\left[2a_r - \frac{\mu(r_{CO} - r_0)}{r_{CO}^2 r_0}\right]\right\} \tag{7}$$

The semimajor axis of the final orbit is related to the apoapse radius by

$$a = -\left[r_f^2\big/(p - 2r_f)\right] \tag{8}$$

Substituting this relation into the traditional form for orbital energy,

$$\varepsilon = -(\mu/2a) \tag{9}$$

yields an expression for the orbital energy in terms of the final apoapse radius,

$$\varepsilon = \mu(p - 2r_f)/2r_f \tag{10}$$

Following Battin's[4] notation, it is convenient to use the dimensionless radial thrust parameter, which is related to the radial thrust acceleration by

$$a_r = \mu\big/8r_0^2\beta^2 \tag{11}$$

When the orbital energy in Eq. (10) is equated with the energy immediately after cutoff in Eq. (7), an expression for the cutoff radius can be derived in terms of the initial radius, final radius, and the normalized radial thrust parameter:

$$r_{CO} = r_0 + 4r_0(r_f - r_0)^2\beta^2\big/r_f^2 \tag{12}$$

Once the necessary cutoff radius is determined, it is convenient to calculate the time of flight during the thrust period. Rearranging Eq. (3) and substituting the relation for radial thrust acceleration found in Eq. (11) produces Eq. (13) for the elapsed time during a thrust period,

$$\sqrt{\frac{\mu}{r_0^3}}(t_{CO} - t_0) = \frac{2\beta}{\sqrt{r_0}}\int_{r_0}^{r_{CO}} \frac{r\,dr}{\sqrt{(r - r_0)\left(r^2 - 4\beta^2 r_0(r - r_0)\right)}} \tag{13}$$

A series of substitutions and manipulations will allow for a closed-form expression for the integral on the right-hand side of Eq. (13). The procedure parallels that originally performed by Battin[4] to get the time of flight information for the constant radial thrust mission profile. Here, the limit of integration is the thrust cutoff radius rather than the apoapse radius used by Battin.[4]

Canonical units are used to make the analysis easier and applicable to a range of problems for varying central bodies and initial orbits. Note that with the assumption of an initial circular orbit,

$$p = r_0 = 1 \tag{14}$$

Also, from the canonical unit definition, $\mu = 1$. The change of variable

$$r = r_0 + z^2 \tag{15}$$

is used to turn the radicand into a fourth-degree polynomial with no odd powers. Using the notation I for the left hand of Eq. (13) produces the relation

$$I = \frac{4\beta}{\sqrt{r_0}} \int_0^{z_1} \frac{r_0 + z^2}{\sqrt{P(z)}}\,\mathrm{d}z \tag{16}$$

The upper limit of the integration then becomes

$$z_1 = [2\beta(r_f - r_0)/r_f]\sqrt{r_0} \tag{17}$$

The polynomial in the radicand has been defined as

$$P(z) = \left(z^2 - \zeta z + r_0\right)\left(z^2 + \zeta z + r_0\right) \tag{18}$$

where

$$\zeta = 2\beta\sqrt{r_0} \tag{19}$$

represents the limit of integration used by Battin[4] for the continuous thrust problem.

Next, the linear fractional transformation is done to reduce $P(z)$ to quadratic factors with no linear terms. Define

$$z = [(1-y)/(1+y)]\sqrt{r_0} \tag{20}$$

to obtain

$$P(z) = \left[4r_0^2\big/(1+y)^4\right]Q(y) \tag{21}$$

with

$$Q(y) = [(1+\beta)y^2 + (1-\beta)][(1-\beta)y^2 + (1+\beta)] \tag{22}$$

The resulting form of the integral is

$$I = 8\beta \int_{y_0}^{1} \frac{1+y^2}{(1+y)^2} \times \frac{\mathrm{d}y}{\sqrt{Q(y)}} \tag{23}$$

The lower limit of the integration is then

$$y_0 = \frac{1 - \sqrt{r_{\mathrm{CO}} - 1}}{1 + \sqrt{r_{\mathrm{CO}} - 1}} \tag{24}$$

A series of partial fraction expansions can be performed on the rational portion of Eq. (23), as well as some additional term manipulations and substitutions, which result in the following equation

$$I = 4\beta(1+\beta)\left(1 - \frac{1}{2}k^2\right)\int_{x_0}^{x_1} \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}} - 4\beta(1+\beta)\int_{x_0}^{x_1}\sqrt{\frac{1-k^2x^2}{1-x^2}}\,\mathrm{d}x + 4\beta C_1 + 4\beta(1+\beta)C_2 \tag{25}$$

For more information consult Battin.[4] The integrated portions are

$$C_1 = \frac{\sqrt{Q(y_0)}}{1+y_0} - \frac{\sqrt{Q(1)}}{2} = \frac{\sqrt{\left[1+\beta+(1-\beta)y_0^2\right]\left[1-\beta+(1+\beta)y_0^2\right]}}{1+y_0} - 1 \tag{26}$$

$$C_2 = \frac{\sqrt{\left(x_0^2-1\right)\left[(\beta-1)^2+4x_0^2\right]}}{x_0(1+\beta)} - \frac{\sqrt{\left(x_1^2-1\right)\left[(\beta-1)^2+4x_1^2\right]}}{x_1(1+\beta)} \tag{27}$$

The new limits of integration become

$$x_0 = \sqrt{(1+\beta)/2} \tag{28}$$

and

$$x_1 = \frac{1+\sqrt{r_{\mathrm{CO}}-1}}{\sqrt{2}}\sqrt{\frac{1+\beta}{r_{\mathrm{CO}} + 2\beta\sqrt{r_{\mathrm{CO}}-1}}} \tag{29}$$

The two integral terms remaining in Eq. (25) are standard forms of elliptic integrals. When these elliptic integrals are recognized, the resulting expression for the burn time becomes

$$t_{\mathrm{CO}} - t_0 = -4\beta^2 + 4\beta(\beta-1)\frac{|1-\eta|}{1+\eta}\sqrt{\frac{1-2\eta\beta+\eta^2}{1+2\eta\beta+\eta^2}} + \frac{4\beta}{1+\eta}\sqrt{(1+\eta^2)^2 - 4\beta^2\eta^2} + 4\beta(1+\beta)[E(k,\phi_0) - E(k,\phi_1)] - 4\beta\frac{1+\beta^2}{(1+\beta)^2}[F(k,\phi_0) - F(k,\phi_1)] \tag{30}$$

where

$$\phi_0 = \sin^{-1}\sqrt{\frac{1+\beta}{2}} \tag{31}$$

$$\phi_1 = \sin^{-1}\left[\left(\frac{1+\eta}{2\beta-\eta}\right)\sqrt{\frac{(1+\beta)\,(\eta-2\beta)^2}{2(1+2\beta\eta+\eta^2)}}\right] \tag{32}$$

$$k = \frac{2\sqrt{\beta}}{1+\beta} \tag{33}$$

For simplicity, the substitution

$$\eta = 2(r_f - 1)\beta/r_f \tag{34}$$

was used. The time values resulting from Eq. (30) will be in terms of canonical time units (TU). To obtain the time in more physically meaningful units, note that the period of the initial circular orbit is 2π TU. The derived equations provide a baseline for comparison with the following numerical analysis.

Numerical Results

Numerical analysis is used to investigate using the small off-axis thrust capability of M2P2. Numerical solutions to the constant radial thrust problem and an additional small in-plane thrust angle are found by numerically integrating the equations of motion. To alleviate singularities occurring with classical orbital elements, modified equinoctial orbital elements were used. A description of these elements may be found in Ref. 5. Table 1 shows the physical values used for the numerical examples with radii given in astronomical units (AU).

The additional problem of a 5-deg maximum steering angle permits a 5-deg in-plane thrust angle α from the radial direction.[6] For all of the optimal trajectories, a nonlinear programming method (NLP), the MATLAB® fmincon, was used in conjunction with the numerical integration of the equations of motion using a Dormand–Prince pair Runge–Kutta integrator. MATLAB's fmincon finds a minimum of a scalar function of several variables given an initial

Table 1 Physical constants

Constant	Value
Initial mass	100 kg
Constant radial thrust	1.0 N
M2P2 power level	1 kW
M2P2 mass flow	0.5 kg/day
Initial orbit radius	1.0 AU
Mars target orbit radius	1.5237 AU
Jupiter target orbit radius	5.2028 AU
Saturn target orbit radius	9.5388 AU

Table 2 Results of numerical solutions

	Mars		Jupiter		Saturn	
Variable	0.0[a]	5.0[a]	0.0[a]	5.0[a]	0.0[a]	5.0[a]
r_{CO}, AU	1.034 (1.035)	1.026	1.176 (1.193)	1.146	1.214 (1.238)	1.180
v_{CO}, km/s	30.562	31.112	34.640	35.279	35.842	36.450
Δv_{circ}, km/s	4.581	4.069	7.333	6.951	6.521	6.286
t_B, day	11.521 (11.438)	10.014	26.163 (28.366)	23.614	28.810 (31.720)	26.189
t_T, yr	0.447	0.478	2.358	2.406	5.567	5.611

[a] α, deg.

estimate of each variable subject to constraints. For the first three cases, Earth–Mars, Earth–Jupiter, and Earth–Saturn, the goal of the NLP is to minimize the impulsive velocity change required to circularize the orbit at the various target planets by varying the burn time and choosing a constant thrust angle. For all cases, the thrust angle converged to the maximum allowed thrust angle of 5 deg. This is because the required velocity change to circularize the final orbit is reduced when any tangential acceleration is applied when compared to the pure radial thrust problem. Table 2 summarizes the results of the three example trajectories. The values in parentheses are the results of the analytical equations. The differences between the numerical and analytic results are due to mass flow being neglected in the analytical derivation. This effect is compounded for trajectories that reach farther out from the starting orbit.

In addition to these cases, a Pluto flyby mission was considered. In this case, the goal of the NLP was to minimize total flight time subject to a hyperbolic excess speed v_∞ of less than 20 km/s at Pluto. The first option considered was a direct transfer to Pluto starting from a circular orbit at 1 AU. This resulted in a cutoff radius of 1.428 AU after 40.461 days of burn and a total flight time of 5.953 years with a $v_\infty$of 20 km/s at Pluto. The second option considered was a burn to Venus, gravity assist at Venus, and then a burn to flyby Pluto. This was done at a close approach altitude of 300 km at Venus, and the resulting velocity vector was calculated using vector geometry and a total turning angle of 122 deg. This resulted in a burn time during the Earth–Venus transfer of 12.786 days with a cutoff radius of 1.043 AU and a flight time of 1.119 years. The second phase of the trajectory had a cutoff radius of 1.267 AU after 45.424 days of burn and an additional flight time of 6.087 years with a v_∞ of 20 km/s at Pluto. Even though the flight time and burn time are longer for the Venus–Pluto portion of the trajectory, the cutoff radius is smaller, which means the mission is more feasible in terms of using solar power to provide the power needed to operate M2P2.

Conclusions

Several outer planet missions were investigated using constant radial thrust propulsion and gravity assist trajectories. For the constant radial thrust missions, it was shown that a 5-deg thrust angle reduced the required impulsive velocity change to circularize the final orbit when compared to the pure radial thrust problem (0-deg thrust angle) and did so without a significant increase in burn time or total flight time. In addition, results indicate that a direct Earth–Pluto transfer was possible, but a Earth–Venus (flyby)–Pluto transfer produced a smaller cutoff radius than the direct transfer, thereby possibly suggesting a solar-powered propulsion system option. Further analysis using outer or multiple planet gravitational assists may provide a larger range of missions deemed relevant for radial thrust propulsion concepts.

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Vehicle Motion Planning with Time-Varying Constraints

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Introduction

WITH the growing emphasis on vehicle autonomy, the problem of planning a trajectory in an environment with obstacles has become increasingly important. This task has been of particular interest to roboticists and computer scientists, whose primary focus is on kinematic motion planning.[1] Typical kinematic planning methods fall into two main categories, roadmap methods and incremental search methods, both of which find collision-free paths in the state space. Roadmap methods generate and traverse a graph of collision-free connecting paths spanning the state space, whereas incremental search methods, including dynamic programming[2] and potential field methods,[3] perform an iterative search to connect the initial and goal states. For the purely geometric path-planning problem, deterministic algorithms have been created that are complete, that is, they will find a solution if and only if one exists. Unfortunately, these suffer from high computational costs, which are exponential in system degrees of freedom. This cost has motivated the development of iterative randomized path-planning algorithms that are probabilistically complete, that is, if a feasible path exists, the probability of finding a path from the initial to final conditions converges to one as the number of iterations goes to infinity. The introduction of the rapidly exploring random trees (RRTs) of LaValle and Kuffner[4] allowed both for computationally efficient exploration of a complicated space as well as incorporation of system dynamics.

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